Cardinality Comparison and a Basic Fact about Subsets of a Set

Basic Fact: if $A_1 \subseteq A$ and $\exists$ f : $A \mapsto A_1$ that is 1-1 (but maybe not onto) , then $\exists$ h: $A \mapsto A_1$ with h being one-to-one and onto.

Proof: Set $A_0 = A$ and define inductively $A_{j+2} = f(A_j)$. Note that $A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \ldots$
Since f is 1-1, it follows that f: $A_j - A_{j+1} \mapsto A_{j+2} - A_{j+3}$ is both one-to-one and onto: it is onto by definition.

Now define h as follows:

If $x \in A_j - A_{j+1}$, j even, then set h(x)=f(x).

If $x \in A_j - A_{j+1}$, j odd, then set h(x)=x.

If $x \in \bigcap_{j=1}^{\infty} A_j$, then h(x)=x .

Since $A = ((A_0 - A_1) \cup (A_1 - A_2) \cup \ldots) \cup (\bigcap_{j=1}^{\infty} A_j)$, this defines h on all of A. It is straightforward to check that h is one-to-one and onto. □

Corollary: If $\exists$ f:$A \mapsto B$, one-to-one, and $\exists$ g: $B \mapsto A$, one-to-one, then $\exists$ k: $A \mapsto B$ one-to-one and onto.

Proof: The composition $g \circ f$: $A \mapsto g(B) \subset A$ is one-to-one since it is the composition of one-to-one maps. By the Basic Fact, there is an onto, one-to-one map h: $A \mapsto g(B)$. Then $k = g^{-1} \circ h$ is one-to-one and onto. □

Now we define $\#A \leq \#B$ ("A has a not larger cardinality than B") to mean $\exists$ f: $A \mapsto B$, f is one-to-one. This makes sense in that, by the Corollary, $\#A \leq \#B$ and $\#B \leq \#A$ implies that $\#A = \#B$ (in the usual sense that $\exists$h: $A \mapsto B$, h one-to-one and onto) . So cardinal numbers can be compared (Exercise: if $\#A \leq \#B$ and $\#B \leq \#C$ then $\#A \leq \#C$). (Exercise: Define $\#A \leq \#B$ and check logical properties).